

# Fillmore Theorem for integers <sup>\*†‡</sup>

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## Abstract

Fillmore Theorem says that if  $A$  is a nonscalar matrix of order  $n$  over a field  $\mathbb{F}$  and  $\gamma_1, \dots, \gamma_n \in \mathbb{F}$  are such that  $\gamma_1 + \dots + \gamma_n = \text{tr } A$ , then there is a matrix  $B$  similar to  $A$  with diagonal  $(\gamma_1, \dots, \gamma_n)$ . Fillmore proof works by induction on the size of  $A$  and implicitly provides an algorithm to construct  $B$ . We develop an explicit and extremely simple algorithm that finish in two steps (two similarities), and with its help we extend Fillmore Theorem to integers (if  $A$  is integer then we can require to  $B$  to be integer).

## 1 Introduction

An *inverse problem* asks for the existence of a matrix with prescribed structural and spectral constraints. The following is an early inverse problem result stated by Fillmore in 1969.

**Theorem 1.** *Let  $A$  be a nonscalar matrix of order  $n$  over a field  $\mathbb{F}$  and let  $\gamma_1, \dots, \gamma_n \in \mathbb{F}$  such that  $\gamma_1 + \dots + \gamma_n = \text{tr } A$ . Then there is a matrix similar to  $A$  with diagonal  $(\gamma_1, \dots, \gamma_n)$ .*

The proof given in [1] is by induction on the size of  $A$  and implicitly provides an algorithm to construct a matrix similar to  $A$  with diagonal  $(\gamma_1, \dots, \gamma_n)$ . Though the algorithm is elementary, it requires some tedious calculus for each induction step. For completeness we will include this proof, remaining as faithful as possible to Fillmore presentation. As the original proof has some inaccuracy then we will incorporate some modifications taken from Zhan [2, Theorem 1.5].

**Lemma 2.** *Let  $A$  be a nonscalar matrix of order  $n \geq 3$  over a field  $\mathbb{F}$  and let  $\gamma \in \mathbb{F}$ . Then there is a nonsingular  $P$  such that  $PAP^{-1} = \begin{bmatrix} \gamma & * \\ * & A_1 \end{bmatrix}$  where  $A_1$  is nonscalar of order  $n - 1$ .*

*Proof.* Since  $A$  is nonscalar, there is a vector  $x$  such that  $x$  and  $Ax$  are linearly independent. Such a vector  $x$  can be taken to be a standard basis vector if  $A$  is not diagonal, or the sum of a pair of standard basis vectors otherwise. Let  $\{x, Ax - \gamma x, x_3, \dots, x_n\}$  be a basis of  $\mathbb{F}^n$ . If  $(\alpha_{ij})_{i,j=1}^n$  is the matrix of  $A$  in this basis, then  $\alpha_{11} = \gamma$ ,  $\alpha_{21} = 1$  and  $\alpha_{31} = \dots = \alpha_{n1} = 0$ . Let  $P = (p_{ij})_{i,j=1}^n$  be equal to the identity except on its entry  $p_{13} = 1 - \alpha_{23}$ . Then  $PAP^{-1}$  has  $\gamma$  on its  $(1, 1)$  entry and 1 on its  $(2, 3)$  entry.  $\square$

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**Proof of Fillmore Theorem:** The proof is by induction on the size of  $A$ . The conclusion of Lemma 2 allows it to be applied repeatedly, leaving the case  $n = 2$  of the theorem for consideration. Again let  $x$  be a vector such that  $x$  and  $Ax$  are linearly independent. Then  $\{x, Ax - \gamma x\}$  form a basis, and in this basis the diagonal of  $A$  is  $(\gamma, \text{tr } A - \gamma)$ .  $\square$

## 2 Alternative algorithm

We present a two steps algorithm that minimizes the required computation. Namely, it starts with  $A$  and performs two similarities to reach a matrix with diagonal  $(\gamma_1, \dots, \gamma_n)$ . On each step the matrix that performs the similarity differs from the identity by one line (row or column). Some results, which can be demonstrated by routine check, are necessary.

**Lemma 3.** Let  $A = (a_{ij})_{i,j=1}^n$  be a nonscalar diagonal matrix over a field  $\mathbb{F}$ , let  $s$  such that  $a_{11} \neq a_{ss}$ , and let  $B = (b_{ij})_{i,j=1}^n$  be equal to the identity except on its entry  $b_{1s} = \frac{1}{a_{ss} - a_{11}}$ . Then  $BAB^{-1}$  is equal to  $A$  except on its entry  $(1, s)$  that is equal to 1.

**Lemma 4.** Let  $A = (a_{ij})_{i,j=1}^n$  be a non-diagonal matrix over a field  $\mathbb{F}$ , let  $r \neq s$  such that  $a_{rs} \neq 0$ , and let  $B = (b_{ij})_{i,j=1}^n$  be equal to the identity except on its row  $s$  where  $b_{sr} = 0$ ,  $b_{ss} = a_{rs}$ , and  $b_{sk} = a_{rk} - 1$  for  $k \notin \{r, s\}$ . Then all off-diagonal entries of the row  $r$  of  $BAB^{-1}$  are equal to 1.

**Lemma 5.** Let  $A = (a_{ij})_{i,j=1}^n$  be a matrix over a field  $\mathbb{F}$  such that for some  $r$  the off-diagonal entries of row  $r$  are equal to 1, let  $\gamma_1, \dots, \gamma_n \in \mathbb{F}$  with  $\gamma_1 + \dots + \gamma_n = \text{tr } A$ , and let  $B = (b_{ij})_{i,j=1}^n$  be equal to the identity except on its column  $r$  where  $b_{rr} = 1$ , and  $b_{kr} = \gamma_k - a_{kk}$  for  $k \neq r$ . Then the diagonal of  $BAB^{-1}$  is  $(\gamma_1, \dots, \gamma_n)$ .

**Proof of Fillmore Theorem:** If  $A$  is nondiagonal then we construct a matrix similar to  $A$  with diagonal  $(\gamma_1, \dots, \gamma_n)$  by the successive application of Lemma 4 and Lemma 5. If  $A$  is diagonal nonscalar then first apply Lemma 3 to obtain a nondiagonal matrix.  $\square$

**Example 6.** We wish to construct a matrix with diagonal  $(3, 5, -2, 6, -1)$  and similar to

$$A = \begin{pmatrix} 4 & 0 & 4 & -3 & 5 \\ 2 & 3 & 0 & 2 & 3 \\ 0 & -2 & 2 & 5 & 4 \\ 7 & 1 & 3 & 4 & 0 \\ 2 & 5 & 3 & 0 & -2 \end{pmatrix}$$

The trace condition is satisfied. We apply Lemma 4 to  $A$  with respect to  $a_{34} = 5$ , so

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -3 & 0 & 5 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \implies B_1 A B_1^{-1} = \begin{pmatrix} \frac{17}{5} & -\frac{9}{5} & 4 & -\frac{3}{5} & \frac{34}{5} \\ \frac{12}{5} & \frac{21}{5} & 0 & \frac{2}{5} & \frac{8}{5} \\ 1 & 1 & 2 & 1 & 1 \\ \frac{172}{5} & \frac{106}{5} & 20 & \frac{17}{5} & -\frac{151}{5} \\ 2 & 5 & 3 & 0 & -2 \end{pmatrix}$$

And we apply Lemma 5 to  $B_1 A B_1^{-1}$  with respect to its third row, so

$$B_2 = \begin{pmatrix} 1 & 0 & -\frac{2}{5} & 0 & 0 \\ 0 & 1 & \frac{4}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{13}{5} & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \implies B_2 (B_1 A B_1^{-1}) B_2^{-1} = \begin{pmatrix} 3 & -\frac{11}{5} & \frac{59}{25} & -1 & \frac{32}{5} \\ \frac{16}{5} & 5 & -\frac{171}{25} & \frac{6}{5} & \frac{13}{5} \\ 1 & 1 & -2 & 1 & 1 \\ 37 & \frac{119}{5} & \frac{824}{25} & 6 & -\frac{138}{5} \\ 3 & 6 & -\frac{1}{5} & 1 & -1 \end{pmatrix} \quad \square$$

### 3 Fillmore Theorem for integers

In Example 6, if we start applying Lemma 4 to  $A$  with respect to  $a_{42} = 1$  and after that we apply Lemma 5 to the resulting matrix with respect to its fourth row we obtain the sequence

$$A \Rightarrow \begin{pmatrix} 4 & 0 & 4 & -3 & 5 \\ 60 & -6 & 37 & -6 & 37 \\ 12 & -2 & 6 & 5 & 2 \\ 1 & 1 & 1 & 4 & 1 \\ -28 & 5 & -7 & 0 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & -1 & 3 & 47 & 4 \\ 71 & 5 & 48 & 630 & 48 \\ 4 & -10 & -2 & 47 & -6 \\ 1 & 1 & 1 & 6 & 1 \\ -32 & 1 & -11 & -151 & -1 \end{pmatrix}$$

This last matrix is integer, similar to  $A$ , and has diagonal  $(3, 5, -2, 6, -1)$ . In order to obtain an integer matrix it was important that  $A$  had an off-diagonal entry equal to 1.

**Lemma 7.** *Let  $A = (a_{ij})_{i,j=1}^n$  be a nonscalar integer matrix. Then there is an integer matrix similar to  $A$  with an off-diagonal entry equal to 1.*

*Proof.* If  $A$  is diagonal then the result follows from Lemma 3, and if  $A$  is nondiagonal and has a nonzero entry equal to 1 then there is nothing to prove. So suppose that  $A = (a_{ij})_{i,j=1}^n$  is nondiagonal with none of its off-diagonal entries equal to 1. Without loss of generality assume that  $a_{12} \neq 0$ . We develop a simple algorithm:

**Step 1.** If  $(a_{13}, \dots, a_{1n}) \neq (0, \dots, 0)$  then go to Step 2. Otherwise go to Step 3.

**Step 2.** Let  $k$  be the minimum integer of  $\{3, 4, \dots, n\}$  such that  $a_{1k} \neq 0$ . Let  $m = \gcd(a_{12}, a_{1k})$ ,  $p = \frac{a_{12}}{m}$ , and  $q = \frac{a_{1k}}{m}$ . As  $p$  and  $q$  are coprimes then let  $r$  and  $s$  two integers such that  $ps - qr = 1$ . Let  $B = (b_{ij})_{i,j=1}^n$  be equal to the identity except on its entries  $b_{22} = p$ ,  $b_{2k} = q$ ,  $b_{k2} = r$ , and  $b_{kk} = s$ . Then  $BAB^{-1}$  is integer and its first row is equal to  $(a_{11}, m, 0, \dots, 0, a_{1,k+1}, \dots, a_{1n})$ . If  $m = 1$  then we have finished. Otherwise do  $A := BAB^{-1}$  and go to Step 1.

**Step 3.** Let  $B = (b_{ij})_{i,j=1}^n$  be equal to the identity except on its entry  $b_{11} = 1/a_{12}$ . Then  $BAB^{-1}$  is an integer matrix with first row  $(a_{11}, 1, 0, \dots, 0)$ . And we have finished. □

**Theorem 8.** *Let  $A$  be a nonscalar integer matrix of order  $n$  and let  $\gamma_1, \dots, \gamma_n \in \mathbb{Z}$  such that  $\gamma_1 + \dots + \gamma_n = \text{tr } A$ . Then there is an integer matrix similar to  $A$  with diagonal  $(\gamma_1, \dots, \gamma_n)$ .*

*Proof.* By Lemma 7  $A$  is similar to an integer matrix  $A_1$  with an off-diagonal  $(r, s)$  entry equal to 1. Applying Lemma 4 to  $A_1$  with respect to its  $(r, s)$  entry we obtain a similar matrix  $B_1 A_1 B_1^{-1}$  which is also integer (since  $B_1$  is integer and  $\det B_1 = 1$ ) and has the off-diagonal entries of its row  $r$  equal to 1. Applying Lemma 5 to  $B_1 A_1 B_1^{-1}$  with respect to its row  $r$  then we obtain a similar matrix  $B_2 (B_1 A_1 B_1^{-1}) B_2^{-1}$  which is also integer (since  $B_2$  is integer and  $\det B_2 = 1$ ) and has diagonal  $(\gamma_1, \dots, \gamma_n)$ . □

## References

- [1] P. A. Fillmore, On similarity and the diagonal of a matrix, Amer. Math. Monthly 76 (1969) 167–169.
- [2] X. Zhan, Matrix Theory, Graduate Studies in Mathematics 147, American Mathematical Society, 2013.